6 - 12 Tasks - Possible Solutions and Notes

Quarterly Meeting # 2
November 2013
At the beginning of 6th grade, friends Jennifer and Susan decided to read the books within the Series of Unfortunate Events. Jennifer read a total of 6 books over 8 months. Susan read a total of 8 books over 10 months.

a. Which girl is reading at a faster rate? Make a table or graph to justify your answer.

b. At this rate, how many books will Jennifer read after one year? How many books will Susan read? Explain how you arrived at your answers.

Teacher Notes:
- This is an instructional task where the teacher should be looking for multiple solution paths from the students and expecting to hear rate language as the students are asked to explain their thinking and/or reasoning.
- Care should be taken to focus on unit rate, and a good discussion could be built around the do they know how to determine it from a table, graph or equation.

<table>
<thead>
<tr>
<th>College- and Career-Ready Standards for Mathematical Content</th>
<th>Standards for Mathematical Practice</th>
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</thead>
<tbody>
<tr>
<td>Understand ratio concepts and use ratio reasoning to solve problems.</td>
<td>1. Make sense of problems and persevere in solving them.</td>
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<tr>
<td>1. Understand the concept of a ratio, and use ratio language to describe a ratio relationship between two quantities. [6-RP1] Examples: “The ratio of wings to beaks in the bird house at the zoo was 2:1 because for every 2 wings there was 1 beak.” “For every vote candidate A received, candidate C received nearly three votes.”</td>
<td>2. Reason abstractly and quantitatively.</td>
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<tr>
<td>2. Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. [6-RP2] Examples: “This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $\frac{3}{4}$ cup of flour for each cup of sugar.” “We paid $75 for 15 hamburgers, which is a rate of $5$ per hamburger.” (Expectations for unit rates in this grade are limited to non-complex fractions.)</td>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
</tr>
<tr>
<td>3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. [6-RP3] a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. [6-RP2a]</td>
<td>4. Model with mathematics.</td>
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<td>5. Use appropriate tools strategically.</td>
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<td>6. Attend to precision.</td>
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<td>7. Look for and make use of structure.</td>
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<td>8. Look for and express regularity in repeated reasoning.</td>
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## Essential Understandings

- A ratio can compare a part of a quantity to the whole (part-whole), or to another part (part-part). A third type of ratio compares two different things, e.g., miles/hour.
- Reasoning with ratios involves attending to and coordinating two quantities. *(NCTM EU #1)*
- A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit *(NCTM EU #2)*

## Explore Phase

### Possible Solution Paths

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<tbody>
<tr>
<td>a.</td>
<td>CONVERGING TO UNIT RATES AND COMPARING</td>
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<tr>
<td></td>
<td>Jennifer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 books</td>
<td>=</td>
<td>3 books</td>
<td>=</td>
<td>.75 book</td>
</tr>
<tr>
<td>8 months</td>
<td>4 months</td>
<td>1 month</td>
<td>1 month</td>
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</table>

Susan

| 8 books | = | 4 books | = | .8 book | or | 4/5 book |
| 10 months | 5 months | 1 month | 1 month |       |

Susan is reading at a faster rate because 4/5 > 3/4

### Assessing and Advancing Questions

**Assessing:**
- How can you explain your work?
- Why did you choose to set your problem up as a ratio?
- How did you get from this ratio to the next?
- How did you get a decimal answer?
- How do you know Susan is reading at a faster rate?

**Advancing:**
- How are you going to use this and put it into a table or a graph?
- How do you know 4/5 > ¾? Can you prove it using a picture?
b. **MAKING TABLES**

### Jennifer

<table>
<thead>
<tr>
<th>Books</th>
<th>Months</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
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<tr>
<td>9</td>
<td>12</td>
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<tr>
<td>12</td>
<td>16</td>
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<tr>
<td><strong>15</strong></td>
<td><strong>20</strong></td>
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</tbody>
</table>

### Susan

<table>
<thead>
<tr>
<th>Books</th>
<th>Months</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
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<tr>
<td>8</td>
<td>10</td>
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<tr>
<td>12</td>
<td>15</td>
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<tr>
<td><strong>16</strong></td>
<td><strong>20</strong></td>
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</tbody>
</table>

If both girls continue at a constant rate, then after 20 months Jennifer will have only read 15 books and Susan has read 16 books. Therefore, Susan is reading at a faster rate.

**Scaling:** Decreasing or increasing both sides by the same amount (Divide each side by two, or double each side)

### Jennifer

<table>
<thead>
<tr>
<th>Books</th>
<th>Months</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>8</td>
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<tr>
<td>3</td>
<td>4</td>
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<tr>
<td>9</td>
<td>12</td>
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<tr>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td><strong>15</strong></td>
<td><strong>20</strong></td>
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### Susan

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<thead>
<tr>
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<tr>
<td>8</td>
<td>10</td>
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<tr>
<td>4</td>
<td>5</td>
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<tr>
<td>12</td>
<td>15</td>
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<tr>
<td><strong>16</strong></td>
<td><strong>20</strong></td>
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</table>

**Assessing:**

- How did you get to 3 books?
- Why did you go all the way to 15 books? Did you need to?
- How do you know Susan is reading at a faster rate?

**Advancing:**

- Does Susan read a book every month?
- Can you determine how many months it takes Susan to read one book?
- What would the graph of each girl’s line look like?

**Assessing:**

- Can you explain to me how you got these numbers on your table?
- How did you know when you had enough data in your table to derive at your answer?

**Advancing:**

- Could you continue this pattern forever?
- Can you reduce these to a unit rate? How?
It takes Jennifer 16 months to read 12 books and takes Susan 15 months to read 12 books, therefore Susan is reading at a faster rate.

Or, Jennifer can read 15 books in 20 months, where Susan will read 16 books in 20 months, therefore Susan is reading at a faster rate.

**GRAPHING**

b. 

Susan, shown as the red line, is reading at a faster rate because her line is steeper.

**Assessing:**
What do these lines represent?
How did you know where to draw the lines?

**Advancing:**
If you reversed the axis and put books on the x-axis and months on the y-axis, would you still derived at the same answer? Justify your response.
How would that have changed the graph of the lines? Would Susan’s still be steeper?
How can we determine the unit rate for the lines that you graphed?
Susan (red line) is reading more books than Jennifer (blue line) each month, therefore she is reading at a faster rate.

Assessing:
How do you read this graph?
How can you tell that Susan is reading more books than Jennifer?

Advancing:
Could you have made your chart and used the months for the x-axis and books for the y-axis?
How would that change how the lines graphed would look?
How can you use the graph to help you determine the unit rate?
c.

PROPORTIONS

<table>
<thead>
<tr>
<th></th>
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<th>Susan</th>
</tr>
</thead>
<tbody>
<tr>
<td>books</td>
<td>.75 books</td>
<td>.8 books</td>
</tr>
<tr>
<td>months</td>
<td>1 month</td>
<td>1 month</td>
</tr>
<tr>
<td></td>
<td>12 months</td>
<td>12 months</td>
</tr>
</tbody>
</table>

\[
\frac{.75 \text{ books}}{1 \text{ month}} = \frac{x}{12 \text{ months}} \quad \frac{.8}{1 \text{ month}} = \frac{x}{12 \text{ months}}
\]

\[X = 9 \text{ books} \quad x = 9.6 \text{ books}\]

RATIO BOXES

<table>
<thead>
<tr>
<th></th>
<th>Jennifer</th>
<th>Susan</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>given rate</td>
<td>desired rate</td>
</tr>
<tr>
<td>Books</td>
<td>6</td>
<td>x</td>
</tr>
<tr>
<td>Months</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

\[8x = 72 \quad \frac{8 = x}{10} \quad 10x = 96 \]

\[x = 9 \text{ books} \quad x = 9.6 \text{ books}\]

Assessing:
Can you explain to me why you set up your problem this way?
What does the “x” represent?
How does your answer relate back to the question?

Advancing:
Did you have to use the unit rate to solve this problem?
Can you tell me how many books each girl would read in 2 years? 3 ½ years?

Assessing:
Why did you choose to use a ratio box?
What ratio was given in the problem?
Can you explain to me how you came up with your ratio box?
What does the “x” represent in your problem?
How did you use your ratio box to set up your proportion?
How does your answer relate back to the question?

Advancing:
Could you have set up your ratio box differently?
Can you tell me how many books each girl will read in 2 years? 3 ½ years?
Can you make a graph to illustrate this information?
Reading the graph at 12 months, Jennifer will be at 9 books, and Susan will be a little more, but it does not give exact amounts. Students should be pushed to precision for this answer using another method.

Assessing:
What does your graph tell you?
According to the graph, can you tell me exactly how many books Jennifer will read in 12 months? Susan?
Can you estimate how many books it appears that Susan can read in 12 months? Explain.

Advancing:
How can you determine exactly how many books that Susan read in 12 months?
Explain how you can determine the unit rate from graph?
Is this the best method to use for precision?
How can you use your information from the graph to make a table?
### Possible Student Misconceptions

Students may have a tendency to confuse the books and the months. As a teacher, make sure that the student is comparing the same unit rates.

Rather than scaling accurately, students may think there is a constant difference of two between the books and the months.

### Assessing:
- Can you explain to me how you set up your problem?
- What is the ratio that is given in the problem?

### Advancing:
- Can you tell me the unit rate in books per month? What about months per book?

### Assessing:
- Can you explain to me what your numbers are referring to?

### Advancing:
- If Jennifer is reading 6 books in 8 months, how can you figure out how many books she is reading in one month?
- What do we call it when you can tell me how many books she reads in one month?

### Entry/Extensions

<table>
<thead>
<tr>
<th>Assessing and Advancing Questions</th>
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<tbody>
<tr>
<td><strong>Assessing:</strong> What does the problem say?</td>
</tr>
<tr>
<td>What information is the problem giving me?</td>
</tr>
<tr>
<td>What could I do to help me organize or set up the problem?</td>
</tr>
<tr>
<td><strong>Advancing:</strong> Can you make a table organize the information?</td>
</tr>
<tr>
<td>Which tool would you like to use to organize this information to compare the rates, a table or graph? Why?</td>
</tr>
</tbody>
</table>

### If students can’t get started....

- **Assessing:** What does the problem say?
- **Advancing:** Can you make a table organize the information?

### If students finish early....

- **Advancing:** Can you determine how many books each girl will read in 2 years? 3 ½ years?
- Susan’s mom will give her $50.00 if she has reads 20 books in 2 years. At this rate, will she earn the $50.00?
<table>
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<th>Discuss/Analyze</th>
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<tbody>
<tr>
<td>Whole Group Questions</td>
</tr>
<tr>
<td>What was the ratio in this problem?</td>
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<tr>
<td>How can I use a ratio to find unknown quantity?</td>
</tr>
<tr>
<td>Who can explain to the class how we would find the unit rate in this problem on the graph?</td>
</tr>
<tr>
<td>How could we find the unit rate from the table?</td>
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<tr>
<td>Is there more than one unit rate?</td>
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<tr>
<td>What does the unit rate mean?</td>
</tr>
<tr>
<td>Why would finding the unit rate help us answer the question?</td>
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<tr>
<td>What is the relationship between the table, graph and the equation?</td>
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<tr>
<td>Can someone pose another problem for the class by providing us a rate to use? (ie. Johnny drove 30 miles in 20 minutes and Stacy drove 24 miles in 15 minutes to get to the movie theater. Which person was driving faster to get to the movies?)</td>
</tr>
</tbody>
</table>
Darien is working on a science project involving the diversity of plants found in a field near his home. To do his science project, he has measured a 12 foot by 15 foot plot of land and has divided this plot into 1 foot by 1 foot squares. He is listing the plants in each square and collecting samples of the plants for his project. After 2 ½ hours, Darien has collected samples from 2/9 of the field.

a) How many 1 foot by 1 foot squares has he completed in 2 ½ hours? Explain your reasoning.

b) If Darien worked at a constant rate, what fraction of the field would he have completed in 1 hour? Draw a picture to support your calculation.

c) If Darien continues to work at the same constant rate, how long will it take him to collect the samples from his entire plot of land? Explain how you know.

d) Darien’s partner LiliAndra is doing a similar investigation in a field near her home. This field is long and narrow, so she has measured a 6 foot by 30 foot plot of land to use and has divided her plot into 1 foot by 1 foot squares. In 3 hours and 20 minutes, LiliAndra has collected samples from 60 of her 1 foot by 1 foot squares. At what rate is LiliAndra collecting samples? How long will it take her to complete her field?

e) Draw a graph to compare Darien’s progress to LiliAndra’s. Does the shape of the field make a difference in your graph? Why or why not?

Teacher Notes:

Some parts of this task require students to report a rate. Note that rates can be given either as the number of squares per hour or as the fraction of the field completed per hour.

Some of the diagrams may be difficult to draw in order to show equal parts. Students may need additional guidance in order to make sense of the diagrams.

Parts (d) and (e) can be omitted in the interest of time if necessary. However, the question regarding the impact of the shape of the field on calculations and mathematical thinking should be considered at some point in the discussions.

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<th>College- and Career-Ready Standards for Mathematical Content</th>
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<tr>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
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Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units. [7-RP1]
   
   Example: If a person walks $\frac{3}{4}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{3}{4}$ miles per hour, equivalently $2$ miles per hour.

2. Recognize and represent proportional relationships between quantities. [7-RP2]
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. [7-RP2a]
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. [7-RP2b]
   c. Represent proportional relationships by equations. [7-RP2c]
      
      Example: If total cost $t$ is proportional to the total number $n$ of items purchased at a constant price $p$, the relationship between the total cost and the number of items can be expressed as $t = pn$.
   d. Explain what a point $(x, y)$ on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0, 0)$ and $(1, p)$ where $p$ is the unit rate. [7-RP2d]

3. Use proportional relationships to solve multistep ratio and percent problems. [7-RP3]
   
   Examples: Sample problems may involve simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, and percent error.

Essential Understandings

- Reasoning with ratios involves attending to and coordinating two quantities.
- Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.
- Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.
- Linear functions have constant rates of change.

Explore Phase

Possible Solution Paths

a) Students can use some calculations to determine how many squares have been completed:
   - Total number of squares: $12 \times 15 = 180$ squares.
   - $2/9$ of the field has been completed, so $2/9 \times 180$ squares = $40$ squares have been completed.

Assessing and Advancing Questions

Advancing Questions:

How many 1 foot by 1 foot squares are in Darien’s plot? How can this help you answer the question?

Can you draw a picture to help you with this problem?
Students can also draw a picture to determine how many squares have been completed. The original 12 foot by 15 foot field can be drawn as a rectangle. The rectangle can be divided into 9 equal parts, with 2 of those 9 parts shaded. The student can then count the number of squares in those 2 parts.

Note that the “2 ½ hours” information is not needed to complete part (a).

b) Students can calculate the number of squares completed per hour using their information from part (a):

\[
\frac{40 \text{ squares}}{2 \frac{1}{2} \text{ hours}} = \frac{40 \text{ squares}}{5/2 \text{ hours}}
\]

\[
= 16 \text{ squares/hour},
\]

and \( \frac{16 \text{ squares per hour}}{180 \text{ squares in the field}} \)

\[
= \frac{4}{45} \text{ of the field per hour.}
\]

Students can also calculate the fraction of the field completed per hour:

\[
\frac{2/9 \text{ field}}{2 \frac{1}{2} \text{ hours}} = \frac{2/9 \text{ field}}{5/2 \text{ hours}}
\]

\[
= \frac{4}{45} \text{ of the field per hour.}
\]

**Assessing Questions:**
- How did you know how many squares were in Darien’s plot of land?
- How did you figure out how many squares had been completed?
- How did you use the 2 ½ hours in your calculations?

**Advancing Questions:**
- What information do you need to build your fraction?
- If you know how much is completed in 2 ½ hours, how can you use this information to figure out how much is completed in 1 hour?
- How can a picture of the field help you with your calculations?

**Assessing Questions:**
- How did you figure out your fraction? What information did you use to complete your calculation?
- Explain your diagram to me. How does your diagram support your calculations?
For the diagram, we will go back to the graph in part (a). Note that the portion highlighted in yellow represents the part of the field Darien has completed.

We need to figure out how much Darien has completed in one hour. Since Darien has worked for 2 ½ hours, we need a way to divide our “completed” portion into equal parts that can be connected in some way to the 2 ½ hours. We know that 2 ½ hours is equal to 5/2 hours, or 5 “1/2 hour” pieces. We begin by dividing the yellow portion into 5 equal parts; these divisions are noted in red.

Since 1 full hour is made up of 2 of the “1/2 hour” pieces, we will shade 2 of the 5 “pieces” in our picture:

![Diagram showing the completed portion and divisions]

Next, we try to extend our “divisions” to the entire field (denoted in red below).
We note that this does not divide the field into equal parts, so we will have to add some subdivisions (in blue on the next graph). Our region shaded in orange, which represents the part of the field completed in 1 hour, is now $\frac{4}{45}$ of the entire field.

c) Using calculations:

- $\frac{1 \text{ field}}{\left( \frac{4}{45} \text{ of the field per hour} \right)} = \frac{45}{4} \text{ hours}.$

Advancing Questions:

If you know how much of the field you can complete in one hour, how can you use this to figure out how long it takes to complete the field?
= 11 ¼ hours to complete the field

OR, using the number of squares:

- 1 field = 180 squares, and 180 squares / (16 squares per hour) = 11 ¼ hours to complete the field.

OR, using a proportion:

- (2/9 of the field) / (5/2 hours) = (1 field) / (x hours).
  
  So 2/9 times x = 5/2 times 1.

  So x = 5/2 divided by 2/9 = 45/4 = 11 ¼ hours.

OR, using a diagram similar to the last one in part (b), we divide the field into 45 equal parts (since we know Darien completes 4/45 of the field per hour), and we circle groups of 4 parts (since 4 “parts” equals 1 hour of work). There are 11 groups of 4 parts (so we have 11 whole hours of work), with 1 part “left over”, and this 1 part represents ¼ of an hour. (Note: In the diagram below, groups of “4 parts” are represented with different colors.)

Can you use the number of completed squares to do your calculation?

Can you use your picture do your calculation?

Assessing Questions:
How did you do your calculation?

Are there any other ways to do the problem? If you use a different method, will you get a different answer?
d) Using calculations:
  3 hours and 20 minutes = 3 1/3 hours =
  10/3 hours, so
  (60 squares) / (10/3 hours) = 18
  squares/hour.

OR, using fractions:
  (60 squares) / (180 squares/field) = 1/3
  of the field completed in 3 1/3 hours,
  so (1/3 field) / (10/3 hours) = 1/10
  field/hour.

OR, using the diagram:
  60 squares are shaded yellow and are
  completed in 10/3 hours. Dividing the
  60 squares into 10 “1/3 hour” segments
  (using the red divisions) gives 6 squares
  per “1/3 hour”, or 18 squares/hour.

To complete the field:

  (180 squares) / (18 squares/hour) =
  10 hours.

OR, using the rate of 1/10 field/hour:
  In 1 hour, 1/10 of the field is completed, so since the entire

---

**Advancing Questions:**

How can you describe the rate at which LiliAndra is completing her work?

What fraction of the field is completed each hour?

Would a diagram be helpful in solving the problem?

**Assessing Questions:**

How did you calculate your rate?

Is there more than one way to describe LiliAndra’s rate?

Explain your calculations to me.
field = 10/10 of the field, it will take 10 hours to complete the field.

OR, using the diagram:
In 3 hours and 20 minutes, 60 squares (or 1/3 of the field) is completed, so it will take 3 x 3 hours and 20 minutes = 10 hours to complete the field.

e) Darien completes 16 squares per hour; after 0 hours he has completed 0 squares and after 1 hour he has completed 16 squares, so we graph the line that goes through the points (0,0) and (1,16) (where the x-coordinate represents the number of hours and the y-coordinate represents the number of squares completed). Darien’s graph is the red line below.

Similarly, LiliAndra completes 18 squares per hour. LiliAndra’s

<table>
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<th>Advancing Questions:</th>
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<tr>
<td>In your graph, what will x represent? What will y represent?</td>
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<tr>
<td>Will your graph representing Darien’s progress be a line or curved? Why?</td>
</tr>
<tr>
<td>How will you determine what the graph should look like?</td>
</tr>
<tr>
<td>(Similar questions can be asked for LiliAndra’s progress.)</td>
</tr>
</tbody>
</table>
graph is in blue.

The shape of the field does not affect the graph since, in both cases, we are counting the number of 1 foot by 1 foot squares completed in each hour.

**Assessing Questions:**

What does the slope of each line represent?

How can you use the graph to determine who finishes first?

What point on the graph shows how long it takes each person to complete the graph?

**Possible Student Misconceptions**

<table>
<thead>
<tr>
<th>In part (a), students may try to multiply 2/9 by 2 ½ in order to use all of the information given in the problem.</th>
<th>What does the 2 ½ represent? Does this help you answer the question?</th>
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<tbody>
<tr>
<td>In part (b), students may find it difficult to draw the picture.</td>
<td>What fraction of the field is completed in 2 ½ hours? Can you draw a picture of this? Can you use your drawing to figure out how much of the field is completed in 1 hour?</td>
</tr>
<tr>
<td>In part (c), students can work using either the number of squares completed in one hour or the fraction of the field completed in one hour.</td>
<td>Can you use a diagram with the number of squares completed every hour to help you determine how long it will take to complete the field?</td>
</tr>
<tr>
<td>In part (d), the rate can be given either using the number of squares</td>
<td>What does the rate measure in this case? How can you use this to</td>
</tr>
</tbody>
</table>
Students may be confused regarding which should be used. Determine LiliAndra’s rate of completion?

**Entry/Extensions**

<table>
<thead>
<tr>
<th><strong>Assessing and Advancing Questions</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>One of the key pieces of information students need to know is how many 1 foot by 1 foot squares are marked off.</td>
</tr>
<tr>
<td>Several parts of the problem can be approached in different ways.</td>
</tr>
</tbody>
</table>

**Discuss/Analyze**

**Whole Group Questions**

- How does the use of a diagram help you with your problems?
- Compare the different solution paths. Some of you gave me rates as the number of squares per hour and others gave me rates as the fraction of the field completed per hour. Are these equivalent? Why or why not?
- Did your different rates result in different graphs in part (e)? Why or why not?
Sally bought a new car. Her total cost including all fees and taxes was $15,000. She made a down payment of $4300. She financed the remaining amount with no interest. Sally is going to pay off the remainder of the loan using equal monthly payments.

a) After 12 monthly payments, Sally has a remaining balance of $7460. How many months will it take for Sally to pay off the loan? Show how you decided.

b) Both of the first quadrant graphs below could be used to represent Sally’s situation. Title, scale, and label the axes on each graph so that the graph makes sense in terms of Sally’s situation. Address key aspects of the graph, such as the intercepts and the slope as they relate to Sally’s situation.

Teacher Notes:
- The teacher may need to explain the statement “She financed the remaining balance with no interest.”
- Question A has two possible answers. It asks how many months will it take her to pay off the loan? Some students will answer this by saying the total number of months, while some will say the number of months past 12. This is an opportune time to talk about justifying your answer based on the assumptions made.
- Another part of question a is the fact that the number of payments does not come out to an exact answer. Students need to discuss that even though Sally is paying $270 per month, the last payment would not be as much.
- In Part B teachers may need to provide graphs with some gridlines if students really struggle. There are some included at the end of the task that you can insert.
<table>
<thead>
<tr>
<th>College- and Career-Ready Standards for Mathematical Content</th>
<th>Standards for Mathematical Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand the connections among proportional relationships, lines, and linear equations. 8. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation y = mx + b for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b. [8-EE6] Define, evaluate, and compare functions. 11. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. (Function notation is not required in Grade 8.) [8-F1]</td>
<td>1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.</td>
</tr>
</tbody>
</table>

Essential Understandings

- Functions provide a tool for describing how variables change together. Using a function in this way is called modeling, and the function is called a model.
- Functions can be represented in multiple ways—in algebraic symbols, situations, graphs, verbal descriptions, tables, and so on—and these representations, and the links among them, are useful in analyzing patterns of change.
- Some representations of a function may be more useful than others, depending on how they are used.
- Linear functions have constant rates of change.

Explore Phase

**Possible Solution Paths**

- a) Students could subtract $4300 from $15000 = $10700
  
  They know that in 12 months the balance went from $10700 to $7460, a difference of $3240. $3240/12 months = $270 per month

  $7460/270 = 27.63 months
  
or
  $10,700/270 = 39.63 months

  (Students could use equations to do the work above)

  $10,700 – 7460 = 12x$ to find the payment. This is the same work done as above, just in equation form.

**Assessing and Advancing Questions**

- **Assessing** – How did you know to subtract $4300 from $15000? How did you come up with $270 per month?

- **Advancing** – How many months total did it take for her to pay off the car? Would she make the $270 payment every month? What does 27.63 months mean?

- **Assessing** – Why did you divide 10700 by 270 and not 7460?

- **Advancing** – How many months total did it take for her to pay off the car? Would she make the $270 payment every month? What does 39.63 months mean?
Assessing – How did you know what the intercepts where? How did you know what the slope is?

Advancing – Can you identify an ordered pair on the graph? If this graph is extended into Quadrant 2, what will the ordered pair be when x = -1? Does this make sense in this problem?

Assessing – How did you know what the intercepts where? How did you know what the slope is?

Advancing – Can you identify an ordered pair on the graph? If this graph is extended into Quadrant 4, what will the ordered pair be when x = 40? Does this make sense in this problem?
Intercepts (0, 10700) Amount left to pay after down payment

(39.63, 0) Number of months to pay off the loan

Slope = -270 this represents the monthly payment. It is negative because the remaining balance goes down each month she makes the payment.

(In either of the 2 graphs above students could do years for the horizontal axis instead of months.)

Assessing – How did you know what the intercepts where? How did you know what the slope is?

Advancing – Can you identify an ordered pair on the graph? What would change about the situation if the y-intercept was 5000?

Intercepts (0, 4300) Amount of down payment

Slope = 270 this represents the monthly payment. It is positive because this adds to the amount of money Sally has paid.

Or
Money Sally has paid for her car

Intercepts (0, 4300) Amount of down payment

Slope = 270 this represents the monthly payment. It is positive because this adds to the amount of money she has paid for the car.

Possible Student Misconceptions

a) Students may blindly divide $7460 by 12 to try and calculate the monthly payment. $7460/12 = $621.67

Assessing - What does the 12 represent? What does 7460 represent?

Advancing - What should we divide by 12 to get the correct monthly payment?

b) Students may get confused on what should go on each axis.

Assessing - What is happening to the amount of money that Sally owes? That she pays?

Advancing - Could a table help the graph make more sense to us? What numbers should we put in our table to help us?

Entry/Extensions

If students can’t get started....

Assessing and Advancing Questions

Assessing – How much was the car? How much money did Sally pay for a down payment? How much does she owe when she drives off the lot? How much did she pay off in the first 12 months?
**Advancing** - How can we use this to help us with how long it will take her to pay off the car?

If students finish early....

**Assessing** – Are these the only two possible graphs?

**Advancing** – Come up with another graph that would work for this situation. Follow the directions for part B.

### Discuss/Analyze

#### Whole Group Questions

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Is the correct answer 27.63 or 39.63 months? Can you round off your answer in this problem? What is the meaning of the decimal part of your answer? What would the last payment be? What if the problem had talked about Sally paying 4% interest on the amount she borrowed?</td>
</tr>
<tr>
<td>b) How can there be a line with positive slope and a line with negative slope that can represent this situation? Can you come up with equations of lines for the two equations? Find a way to verify that these equations are correct.</td>
</tr>
</tbody>
</table>
Task: The Speeding Problem

The city of Cautionville has decided to utilize a new formula for calculating the fine for speeding within their city limits. Speeding violations will be categorized in two ways: regular speeding violations and reckless driving speeding violations. To calculate the charge for each regular speeding violation, the city is enforcing a fee of $60 for each speeding ticket issued. In addition, there will be a charge of $8 for every mile per hour driven that exceeds the city-wide speed limit. The maximum charge for a regular speeding ticket is $300. Anything beyond the $300 amount is considered to be in the reckless driving category and enforces larger penalties and a mandatory court appearance for further potential consequences. The city-wide speed limit is 30 miles per hour.

A. Based on the information above, determine the cost for each of the speeds below. Show how you determined your answers.

   27 miles per hour
   38 miles per hour
   45 miles per hour

B. Write an equation that could be used to determine the total cost of a regular speeding violation ticket. Be sure to define each variable.

   Use your equation to find the cost for a person driving 25 miles per hour over the speed limit. Show your work.

   Use your equation to find the speed that would result in a $212 fine. Show your work.

C. Does your equation represent a function? Why or why not? If it does, what would be the domain and range of the function?

D. At what speeds does the city of Cautionville consider a driver to be reckless? Explain how you determined your answer.
Teacher Notes

This problem is written to have students work with linear equations in a real-world context. Part A is designed to provide an entry point for all students and to indirectly introduce the idea of the domain in context. Some students will need the numeric structures of Part A to develop the equation later. Part B moves students into writing a generalization (equation) for the scenario. It is important to note that there are at least two different equations that represent the scenario. The variable could be defined as strictly the speed a car is going or as the speed driven over the city-wide speed limit. Part C builds into the idea of identifying functions and defining the domain and range based on the context. To address more of IF.A.1, consider introducing or reviewing function notation. Finally, Part D is designed to begin the transition from equations to inequalities.

<table>
<thead>
<tr>
<th>College- and Career-Ready Standards for Mathematical Content</th>
<th>Standards for Mathematical Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create equations that describe numbers or relationships. (Linear, quadratic, and exponential (integer inputs only); for Standard 14, linear only.)</td>
<td>1. Make sense of problems and persevere in solving them.</td>
</tr>
<tr>
<td>12. Create equations and inequalities in one variable, and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. [A-CED1]</td>
<td>2. Reason abstractly and quantitatively.</td>
</tr>
<tr>
<td>14. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities and interpret solutions as viable or non-viable options in a modeling context. [A-CED3]</td>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
</tr>
<tr>
<td>Example: Represent inequalities describing nutritional and cost constraints on combinations of different foods.</td>
<td>4. Model with mathematics.</td>
</tr>
<tr>
<td>Solve equations and inequalities in one variable. (Linear inequalities; literal that are linear in the variables being solved for; quadratics with real solutions.)</td>
<td>5. Use appropriate tools strategically.</td>
</tr>
<tr>
<td>Interpret functions that arise in applications in terms of the context. (Linear, exponential, and quadratic.)</td>
<td>7. Look for and make use of structure.</td>
</tr>
<tr>
<td>29. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.* [F-IF5]</td>
<td>8. Look for and express regularity in repeated reasoning.</td>
</tr>
<tr>
<td>Example: If the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.</td>
<td></td>
</tr>
</tbody>
</table>
### Essential Understandings

- Equations can be used to model real-world scenarios. The variables used in the equation must be defined.
- A function assigns to each element of the domain exactly one element of the range. (HSF – IF.A.1)
- Solutions to an equation must be considered viable based on the domain of the function and context of the scenario.

### Explore Phase

**Possible Solution Paths**

<table>
<thead>
<tr>
<th>Part A.</th>
<th>Assessing and Advancing Questions</th>
</tr>
</thead>
</table>

Students may calculate the speeds whether by adding the fee at the beginning or at the end. Below are a few examples.

**27 mph**
- \[60 + 8(-3) \quad \text{or} \quad 8(-3) + 60\]
- \[60 + 8(27-30) \quad \text{or} \quad 8(27-30) + 60\]
Some may not calculate and justify that there would be no violation since the driver was not speeding.

**38 mph**
- \[60 + 8(8) \quad \text{or} \quad 8(8) + 60\]
- \[60 + 8(38-30) \quad \text{or} \quad 8(38-30) + 60\]

**45 mph**
- \[60 + 8(15) \quad \text{or} \quad 8(15) + 60\]
- \[60 + 8(45-30) \quad \text{or} \quad 8(45-30) + 60\]

### Assessing Questions:

- Describe what each number in your calculations represents. Where is the fee? Where is the cost charged for every mile per hour over the speed limit?
- How did you determine what to multiply the $8 by?
- How do you know that the 27 miles per hour would not generate a ticket? or Why did you not calculate a number for the 27 miles per hour?

### Advancing Questions:

- Would it matter if the $60 fee was added before or after multiplying? Explain.
- How could you write a number sentence that would clearly represent the scenario?

**Part B.**

Students may write different equations based on how the variable is defined. Below are two examples.

### Assessing Questions:

- Describe what each part of your equation represents. What does the 60 represent? the 8? What is the x? the y?
- Talk me through how you solved the problem.
y = 8x + 60 or y = 60 + 8x where x = the speed driven over the speed limit

y = 8(x-30) + 60 or y = 60 + 8(x-30) where x = speed driven

Students will calculate the fine by substituting a value for the domain value of their equation. Examples:

\[ y = 8(25) + 60 \quad \text{or} \quad y = 8(55-30) + 60 \]
\[ = 260 \quad \text{or} \quad = 260 \]

Students will find the speed that will result in a $212 dollar fine by substituting 212 for the range value and solving their equation. Examples:

\[ 8x + 60 = 212 \quad \text{or} \quad 8(x – 30) + 60 = 212 \]
\[ 8x + 60 – 60 = 212 – 60 \quad \text{or} \quad 8x – 240 + 60 = 212 \]
\[ 8x = 152 \quad \text{or} \quad 8x – 180 = 212 \]
\[ 8x ÷ 8 = 152 ÷ 8 \quad \text{or} \quad 8x – 180 + 180 = 212 + 180 \]
\[ x = 19 \quad \text{or} \quad x = 49 \text{ mph} \]
\[ 19 + 30 = 49 \text{ mph} \]

**Part C.**

Students will need to justify whether their equation represents a function or not. Some possible solutions are:

- Some may justify that every domain (speed) has exactly one range (cost of Violation) for the constraints in the scenario.
- Some may graph their equation and show that it meets the Vertical Line Test.

Students will need to define the domain and range. Please note that this does not specify the type of notation that may be used by your curriculum resources. Some possible solutions could be:

- Domain: 30 < x ≤ 60 where x represents the speed driven

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**Advancing Questions:**
- How does your solution relate to the scenario?
- How could you use your number sentences in Part A to help write an equation?
- How could you write an equation that only uses the speed driven? or How could you write an equation that uses the speed driven over the speed limit?

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**Assessing Questions:**
- How do you know that your equation represents a function? Justify your reasoning.
- How did you determine your domain? range?

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**Advancing Questions:**
- At what speed could you first receive a ticket? How does this relate to domain and range?
- How could you use the calculations you did above to help you with this problem?
<table>
<thead>
<tr>
<th><strong>0 &lt; x ≤ 30</strong> where x represents the speed driven over the speed limit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Range:</strong> ( 0 &lt; y ≤ 300 ) where y represents the cost of a ticket</td>
</tr>
<tr>
<td><strong>Domain:</strong> ( 31 ≤ x ≤ 60 ) where x represents the speed driven and integer values</td>
</tr>
</tbody>
</table>

### Part D.

Students may write this in words or mathematical symbols. Below are a few samples:

- **For students who wrote their equation based on speed driven over the speed limit** -
  - Any speeds that exceed 30 mph over the speed limit \( \text{or } x > 30 \)

- **For students who wrote their equation based on the speed driven** –
  - Any speeds that exceed 60 mph \( \text{or } x > 60 \)

### Assessing Questions:
- Show me how your equation and the scenario support your claim for both the domain and range?

### Advancing Questions:
- Could you have speeds that are not integers? How would that affect your domain and range?

### Possible Student Misconceptions

1. **In Part A, students may use the speed driven only and not the speed over the speed limit to calculate the cost of the ticket.**

   Example: \( 8(27) + 60 = 276 \), instead of \( 8(-3) + 60 = 36 \)

   \( 8(38) + 60 = 364 \), instead of \( 8(8) + 60 = 124 \)

2. **Students may use the equation \( y = 8x + 60 \) and assume that the solution to the equation in Part B is the final solution. In this case, x would represent the speed driven over the speed limit, not the actual speed driven. They may not reason to add the 30 mph to the solution.**

### Assessing Questions:
- Talk me through how you solved the problem.
- What does the 27 miles per hour represent? What does the 38 miles per hour represent? How do they relate to the scenario?

### Advancing Questions:
- Would you get a ticket at 27 miles per hour? 38 miles per hour? Explain.
Example: For a ticket cost of $212
\[ 8x + 60 = 212 \]
\[ 8x + 60 - 60 = 212 - 60 \]
\[ 8x = 152 \]
\[ 8x \div 8 = 152 \div 8 \]
\[ x = 19 \]
\[ 19 + 30 = 49 \text{ mph} \]

Advancing Questions:
- I noticed that the solution is 19 mph. Does that make sense in the context of the problem? Explain.
- How does your solution relate to the speeds in Part A?

3. Students may justify their equation to be a function by the Vertical Line Test and not see the domain and range in context.

Assessing Questions:
- How does your graph relate to the scenario?
- What is the domain and range of your graph?

Advancing Questions:
- According to your graph, about how fast would you be going to get a ticket for $380? How many speeds could you drive in order to get a ticket for this amount? Explain.

Entry/Extensions

Assessing and Advancing Questions

Assessing Questions:
- What is the question asking?
- Describe what each number in your calculations represents. Where is the fee? Where is the cost charged for every mile per hour over the speed limit?
- Talk me through how you would find the cost of a ticket.
- How did you determine what to multiply the $8 by?
- How do you know that the 27 miles per hour would not generate a ticket? or Why did you not calculate a number for the 27 miles per hour?

Advancing Questions:
- How could you use your number sentences in Part A to help write an equation?
- What do you notice is changing and what is staying
If students finish early...

Assessing Questions:
- Show me how your equation relates to the scenario.
- Why did you decide to use this equation?
- How do you know that your equation represents a function?
- How does your domain and range relate to the scenario?

Advancing Questions:
- Group or pair students who have different equations – How are your methods similar and different? How are all of your equations related?
- The mayor of Cautionville wants to change the fee for a regular speeding violation, but keep the same maximum cost and criteria (miles per hour over the speed limit) for distinguishing between a regular and reckless violation. Write an equation that would allow this to happen. Mathematically show how your equation fits the mayor’s criteria.

Discuss/Analyze

Whole Group Questions

Select and Sequence refers to when a teacher anticipates possible student strategies ahead of time and then selects and determines the order in which the students’ math ideas/strategies will be shared during the whole group discussion. The purpose of this is to determine which ideas will most likely leverage and advance student thinking about the core math idea(s) of the lesson.

During a whole group discussion, students are sharing their strategies that have been pre-selected and sequenced by the teacher. Strategies to consider sharing in order to advance student thinking are:

- Methods of Calculation in Part A: You may want to consider having students share both calculations (using speed driven and
speed driven over the speed limit) and reasoning about the 27 miles per hour.

- Different Types of Equations and solutions: Share an equation where the variable is defined as the speed driven and one where the variable is defined as the speed driven over the speed limit.
- Justifications of the Function: Share various rationales proving their equation is a function (i.e. graph, table, mapping, etc.)
- Domains and Ranges.

There are lots of rich discussions that need to occur during the Whole Group Share. A discussion around how the 27 miles per hour question relates to the context of the scenario could be a bridge to having students look at the domain. The importance of labeling variables and how the two different equations produce different solutions, but can be used to solve the problem should also be discussed. Part C lends itself to a discussion around the domain and range and what type of numbers would fit the scenario. Finally, Part D could be used to introduce the relationships between equations and inequalities.

Questions to pose during the discussion:

- How did you calculate the cost of each ticket?
- Would you need to calculate the 27 miles per hour? Why or why not?
- How did you use the calculations in Part A to write an equation in Part B?
- How did you define your variables in your equation? Why do the labels matter?
- How does your equation relate to the scenario? Where is each part of your equation in the scenario?
- Do the equations shared have the same solution? Explain.
- Does your equation represent a function? Justify your reasoning.
- How did you define the domain and range of your function? How does your domain and range relate to the scenario?
- Could you receive a ticket for going 30.2 miles per hour? Explain. How does this relate to your domain and range?
- How did you determine the speeds at which a driver is considered reckless?
Task: Comparing Shapes

On a piece of graph paper with a coordinate plane, draw three non-collinear points and label them A, B, C. (Do not use the origin as one of your points.) Connect these points to make a triangle. For each point, take half of the x and y-coordinates and label these new points A’, B’, C’. Connect these points to make another triangle.

1. Compare the distance from the origin to point A’ and from the origin to point A. Do the same for points B’ and B, and for points C’ and C. Describe any relationships you notice.
2. Find the perimeter of triangle ABC and find the perimeter of triangle A’B’C’. Describe any relationships that you notice.
3. Suppose you repeated the directions, but you took a third of the x and y coordinates. Make a conjecture about what would happen to the relationships you noticed in parts 1 and 2.
4. Suppose you repeated the directions, but used a different shape (e.g. quadrilateral, pentagon, hexagon). Make a conjecture about what would happen to the relationships you noticed in parts 1 and 2.
5. Verify your conjectures for numbers 3 and 4.

Extension: If students know how to find the area of non-right triangles, include this after part 2. Students will compare the area of triangle A’B’C’ with the area of triangle ABC. Student should then finish the other parts of the task, making conjectures and proving them, including the areas.

Teacher Notes:
Students will need to know how to find distance between two points in this task.
Because students can place points wherever they want, they should see these relationships will hold in general for any ratio and any shape placed anywhere in the plane.
This task will provide a good foundation for studying similarity and for looking at similarity through dilations from a point.

College- and Career-Ready Standards for Mathematical Content

Use coordinates to prove simple geometric theorems algebraically. (Include distance formula; relate to Pythagorean Theorem.)
32. Find the point on a directed line segment between two given points that partitions the segment in a given ratio. [G-GPE6]
33. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.* [G-GPE7]

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
## Essential Understandings

- The perimeter of figures whose side lengths are in an n:m ratio will also be in an n:m ratio because addition preserves this ratio.
- Behind every proof is a proof idea.
- Empirical verification is an important part of the process of proving, but it can never, by itself, constitute a proof.
- Geometry uses a wide variety of kinds of proofs.

## Explore Phase

### Possible Solution Paths

#### Examining Relationships

Students can place points A, B, C anywhere in the plane.

For example:

- **A**(-2, 6)
- **B**(12, 7)
- **A'**(-1, 3)
- **B'**(6, 3.5)
- **C'(5, -1)
- **C**(10, -2)

The distance from the origin to A is

\[ d = \sqrt{(-2)^2 + 6^2} = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10} \]

The distance from the origin to A' is

\[ d = \sqrt{(-1)^2 + 3^2} = \sqrt{1 + 9} = \sqrt{10} \]

The distance from the origin to A' is half the distance from the origin to A.

The steps would be similar for points B and B', and for points C and C'.

### Assessing and Advancing Questions

#### Assessing Questions

Tell me why you placed your points there.

How did you know where to place points A’, B’, C’?

#### Advancing Questions

How can you find the distance between two points?

How do you calculate perimeter?
Depending on where students placed their points, they might be able to count without using the distance formula, but not being able to use the origin should require them to use it somewhere.

Students should notice that the perimeter of the triangle with the prime points is half of the perimeter of the original triangle.

### Making Conjectures
Students should make a conjecture that if you divide the x and y-coordinates by any number, that number will define the ratio from the origin to the new point and the origin to the original point.

Similarly, students should conjecture the relationships will be maintained with any ratio and the perimeter of any polygon.

### Proving Conjectures
#### Ratios from the Origin
Suppose students divide the x and y-coordinates by n:

Let the coordinates for A be \((r, s)\) and the coordinates for \(a\) be \(\left(\frac{r}{n}, \frac{s}{n}\right)\).

The distance from the origin to A is

\[
d = \sqrt{r^2 + s^2}
\]

The distance from the origin to \(A'\) is

\[
d = \sqrt{\left(\frac{r}{n}\right)^2 + \left(\frac{s}{n}\right)^2} = \frac{1}{n} \sqrt{r^2 + s^2}
\]

The distance from the origin to \(A'\) is \(\frac{1}{n}\) the distance from the origin to A.

(Because this verification is done in general, it will hold for any points B and C. If students use specific points, ask how they can generalize their thinking to account for all points).

#### Ratio of Perimeters
Let the coordinates for A be \((r, s)\) and the coordinates for B be \((p, q)\).

The distance from A to B is

\[
d = \sqrt{(p - r)^2 + (q - s)^2}
\]

---

**Assessing Questions**

Tell me about your conjectures.

**Advancing Questions**

Do you think your conjecture will hold by dividing the coordinates by any number? Why or why not? How could you show this for any value? For any polygon?
The distance from A' to B' is

\[ d = \sqrt{\left(\frac{p-r}{n}\right)^2 + \left(\frac{q-s}{n}\right)^2} = \sqrt{\frac{(p-r)^2 + (q-s)^2}{n^2}} = \frac{1}{n}(p-r)^2 + (q-s)^2 \]

Students can repeat for the distance from A to C, B to C and A' to C', B' to C'. When summing for the perimeter, the \( \frac{1}{n} \) will factor out for triangle A'B'C'.

These ideas hold for any polygon.

**Possible Student Misconceptions**

Students may connect A' to A, B' to B, C' to C and not have an accurate picture for the task.

Assessing Questions

Let's read the problem together. What is it that you need to do first?

Advancing Questions

If you are to divide the coordinates by two, what kinds of numbers might you choose that would make this problem a little easier. Tell me how you think you should use these points to make two triangles.

Students may have a hard time thinking about how to verify their conjectures generally. Students may prove a specific case.

Assessing

Tell me about how you made this conjecture. What do you think is happening? What do you think is happening with your classmates’ relationships?

Advancing

I see you are verifying your conjecture for a quadrilateral you made. How can you verify your conjecture for any quadrilateral?

**Entry/Extensions**

Assessing and Advancing Questions

If students can’t get started....

Assessing Questions

Let’s read the problem together. What is it that you need to do first?

Advancing Questions

Do you remember how to use the distance formula? Show me how to find the distance between these two points.

If students finish early....

Assessing Questions

What did you notice about the relationships in general? What kind of proof did you use?
<table>
<thead>
<tr>
<th>Discuss/Analyze</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole Group Questions</td>
</tr>
<tr>
<td>• What relationships did you notice for the ratio from the origin to the prime points and the original points? Why do you think this is the case?</td>
</tr>
<tr>
<td>• What about the ratio of the perimeters?</td>
</tr>
<tr>
<td>• What can we say if these relationships were true for everyone’s picture in the class, but everyone graphed three different points to start with?</td>
</tr>
<tr>
<td>• How does this make you think about an idea for a proof?</td>
</tr>
<tr>
<td>• Let’s talk about your conjectures. What did you think would happen by dividing the coordinates by a number different than 2? Did anyone try 3 or 4? What effect would this have on the different ratios?</td>
</tr>
<tr>
<td>• I noticed that some of you verified your conjecture with a different case. Remember we just said that everyone in the class could have a different picture but noticed all of the same relationships. We want to be able to verify a conjecture that will work for any case. How can we do that?</td>
</tr>
<tr>
<td>• Why do you think it is important to prove something that will work for any case rather than a specific one?</td>
</tr>
</tbody>
</table>
George has two problems to consider:

**Problem 1:** Erin has moved to a new school in the Virgin Islands. To keep her friends up-to-date on her adventures, she started a blog. In the first week, she had five friends following her blog. Her friends thought the blog was interesting, so the next week each of her friends told two additional friends, who began following the blog. In the next week, each of the new followers from the previous week told two of their friends. This continued for several weeks.

a) Make a three-column table to represent this problem. In the first column, list the number of weeks since Erin’s blog began. In the second column, tell how many NEW followers Erin’s blog has attracted each week. In the last column, give the TOTAL number of followers Erin’s blog has.

b) Write an expression to describe how to find the number of NEW followers Erin's blog has in week N.

c) Write an expression to describe the TOTAL number of followers Erin’s blog has in week N.

**Problem 2:** A frog is sitting a fixed distance away from the pond. He starts hopping towards the pond. In the first hop, he jumps 1/3 of the distance between his original position and the pond. In the second hop, he jumps 1/3 of the remaining distance. In the third hop, he jumps 1/3 of the distance that remains after his second hop. This pattern continues for all of the frog’s hops.

a) Make a table to represent the frog’s progress after each hop. Include a column in your table to show the remaining distance after each hop.

b) Write an expression to describe how to find the remaining distance after the Nth hop.

c) Will the frog ever reach the pond? Why or why not?

George needs your help. Solve both of George’s problems. Describe how these problems are similar and how these problems are different.
This is a long task. Teachers may want to assign problem 1 and problem 2 on different days or assign problem 1 to one half of the class and problem 2 to the other half of the class in order to complete the entire task in a shorter amount of time. If teachers take the second option (half of the class works on each problem), then the similarities/differences of the problems should be part of the class whole-group discussion.

Both of these problems are examples of geometric sequences and geometric series. The first problem illustrates a geometric series that keeps growing without bound—in other words, the geometric series “diverges”. The second problem illustrates a geometric series that grows, but it grows within a bound—in other words, the geometric series “converges”.

<table>
<thead>
<tr>
<th>College- and Career-Ready Standards for Mathematical Content</th>
<th>Standards for Mathematical Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write expressions in equivalent forms to solve problems.</td>
<td>1. Make sense of problems and persevere in solving them.</td>
</tr>
<tr>
<td>14. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.* [A-SSE4]</td>
<td>2. Reason abstractly and quantitatively.</td>
</tr>
<tr>
<td>Example: Calculate mortgage payments.</td>
<td>3. Construct viable arguments and critique the reasoning of others.</td>
</tr>
<tr>
<td></td>
<td>4. Model with mathematics.</td>
</tr>
<tr>
<td></td>
<td>5. Use appropriate tools strategically.</td>
</tr>
<tr>
<td></td>
<td>6. Attend to precision.</td>
</tr>
<tr>
<td></td>
<td>7. Look for and make use of structure.</td>
</tr>
<tr>
<td></td>
<td>8. Look for and express regularity in repeated reasoning.</td>
</tr>
</tbody>
</table>

**Essential Understandings**

- The concept of function is intentionally broad and flexible, allowing it to apply to a wide range of situations. The notion of function encompasses many types of mathematical entities in addition to “classical” functions that describe quantities that vary continuously. For example, matrices and arithmetic and geometric sequences can be viewed as functions.
- Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationships and change.
- The convergence or divergence of the geometric series depends on the value of the common ratio.

**Explore Phase**

**Possible Solution Paths**

**Problem 1: Part (a):** In Week 1, Erin gets her original 5 friends to follow her blog, so the total number of followers is 5.

In Week 2, each of the 5 followers from week 1 tells 2 friends, so the number of new followers is $5 \times 2 = 10$. The total number of followers is then $5 + 10 = 15$.

In Week 3, each of the 10 new followers from week 2 tells 2 friends,
so the number of new followers in week 3 is 10 x 2 = 20. The total number of followers is: 5 (from week 1) + 10 (from week 2) + 20 (from week 3) = 35.

This pattern continues, so your table should look like the following:

<table>
<thead>
<tr>
<th>Week</th>
<th>New Followers</th>
<th>Total Followers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>5 + 10 = 15</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>15 + 20 = 35</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>35 + 40 = 75</td>
</tr>
</tbody>
</table>

**Problem 1: Part (b):** We can use the table in part (a) to establish a pattern.

<table>
<thead>
<tr>
<th>Week</th>
<th>New Followers</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10 = 5 x 2</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20 = 10 x 2 = 5 x 2 x 2 = 5 x 2^2</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>40 = 20 x 2 = 5 x 2 x 2 x 2 = 5 x 2^3</td>
</tr>
</tbody>
</table>

In general, if the student’s table begins with week 1, then the pattern for each of the entries in the “New Followers” column is 5 x 2^{power}. The power on 2 is one less than the week number, so our expression is:

In week N, the number of new followers will be 5 x 2^{N-1}.

(We can write this as f(N) = 5 x 2^{N-1}.)

**Problem 1: Part (c):** Students should recognize that the number of

**Assessing questions:**

What does your variable in the exponent represent? How can you be sure your formula is accurate?

**Advancing questions:**

What happens to the number of new followers each week? How can you use this information to describe a pattern?
new followers is a geometric sequence and the total number of followers can be found using the sum of a geometric series.

**Approach 1:** At a minimum, students should recognize that the sum (assuming that the table in part (a) begins with week 1) is:

\[ 5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1}. \]

**Approach 2:** Once students understand that the total number of followers can be found using the expression in Approach 1, they may either recognize this as the sum of a geometric series and immediately write this using a compact form:

\[
5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1} = \frac{(5 - 5(2)^{N-1})}{(1 - 2)} = 5(2^N) - 5.
\]

**Approach 3:** Students who do not recognize this as the sum of a geometric series can still find the compact form by deriving the formula. To derive the formula, students will need to recognize that if you multiply each term of your sum by 2, you will get the next term in your sum (except when you multiply the last term by 2—then you “create” a new term). You can take advantage of this fact:

Let \( S = 5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1} \)

Then \( 2S = 10 + 20 + 40 + \ldots + 5 \times 2^{N-1} + 5 \times 2^N \)

If we subtract the second equation from the first equation, we have:

\[
S - 2S = 5 - 5(2)^N
\]

or \( -S = 5 - 5(2)^N \)

or \( S = 5(2)^N - 5. \)

**Problem 2: Part (a):** The most difficult part of this problem may be the fact that the distance between the frog and the pond is not given. Student may elect to assign a distance (say, for example, 300 feet) then use the distance they have assigned to begin their

**Assessing questions:**

What effect does not knowing the original distance between the frog and the pond have on your calculations?

How do you know your formula works?

What does the variable represent in your formula?

**Advancing questions:**

What kind of sequence are you examining in this problem? How does knowing the type of sequence in the problem help you?
calculations (so, in the first hop, the frog will have covered 100 feet and have 200 feet remaining). In the solutions provided, the distance between the frog’s initial position and the pond will be 1 unit. This will enable teachers to “adjust” to fit whatever distance the student assigns by multiplying the distances in the solutions by the student’s choice of initial distance.

**Approach 1:** Students may calculate the total distance covered by adding \( \frac{1}{3} \) of the remaining distance to each of the distances covered in the table.

<table>
<thead>
<tr>
<th>Hop</th>
<th>Distance Covered</th>
<th>Distance Left</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{3} + \frac{1}{3} + \frac{2}{3} ) = ( \frac{5}{9} )</td>
<td>( \frac{4}{9} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{5}{9} + \frac{1}{3} + \frac{1}{9} ) = ( \frac{19}{27} )</td>
<td>( \frac{8}{27} )</td>
</tr>
</tbody>
</table>

**Approach 2:** Note that the distance left in the table above forms a nice pattern: the distance left can be found using

\[
\left( \frac{2}{3} \right)^N,
\]

where \( N \) represents the number of hops. Then the distance covered can be calculated using

\[
1 - \left( \frac{2}{3} \right)^N.
\]

This information may be used to fill in the table.

---

**Advancing questions:**

- How far is the frog from the pond? What can you do to work the problem if this distance is not known?
- Would a picture help you understand what is happening in the problem?
**Problem 2: Part (b):** The distance left can be found using
\[
\left(\frac{2}{3}\right)^N,
\]
where \(N\) represents the number of hops. This can be calculated from the table:

<table>
<thead>
<tr>
<th>Hop</th>
<th>Distance Left</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2/3</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{4}{9} = \left(\frac{2}{3}\right)^2)</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{8}{27} = \left(\frac{2}{3}\right)^3)</td>
</tr>
</tbody>
</table>

**Assessing questions:**
What does the variable in your formula represent?

How do you know that your formula works?

**Advancing questions:**
Using your table from part (a), do you see a pattern for the distance left?

How are the numbers in the “distance left” column related to each other?

**Problem 2: Part (c):** Mathematically, the frog will not reach the pond after a finite number of hops. There are several arguments that can be used to support this.

**Argument 1:** The distance between the frog and the pond after \(N\) hops was found in Problem 2, part (b):

The distance left can be found using
\[
\left(\frac{2}{3}\right)^N,
\]
where \(N\) represents the number of hops.

Since there is always a positive distance left, mathematically the frog will never reach the pond.

**Argument 2:** The distance the frog has travelled can be found in three different ways. First, we can use the distance left presented in Problem 2, part (b) above and subtract this distance from 1 (the
total distance that must be covered):

The distance traveled is

$$1 - \left( \frac{2}{3} \right)^N.$$ 

Second, students may recognize that the distance traveled is a geometric series and use the formula for summing a geometric series:

In this geometric series,

$$a_1 = \frac{1}{3}, \quad r = \frac{2}{3},$$

so the sum of the first \( n \) terms is

$$S = \frac{\frac{1}{3} - \left( \frac{2}{3} \right)^N}{1 - \frac{2}{3}} = 1 - \left( \frac{2}{3} \right)^N.$$ 

Third, students may try to generate the sum of the geometric series using a process similar to that in Problem 1, part (c):

Let \( S \) represent the sum of the first \( N \) distances hopped by the frog.

Then

$$S = \frac{1}{3} + \frac{1}{3} \left( \frac{2}{3} \right) + \frac{1}{3} \left( \frac{2}{3} \right)^2 + \frac{1}{3} \left( \frac{2}{3} \right)^3 + \ldots + \frac{1}{3} \left( \frac{2}{3} \right)^{N-1}$$

$$\left( \frac{2}{3} \right) S = \frac{1}{3} \left( \frac{2}{3} \right) + \frac{1}{3} \left( \frac{2}{3} \right)^2 + \frac{1}{3} \left( \frac{2}{3} \right)^3 + \ldots + \frac{1}{3} \left( \frac{2}{3} \right)^N$$

If we subtract the second equation from the first equation, we have:

$$S - \left( \frac{2}{3} \right) S = \frac{1}{3} - \frac{1}{3} \left( \frac{2}{3} \right)^N$$
\( \left( \frac{1}{3} \right)^N = \frac{1}{3} - \frac{1}{3} \left( \frac{2}{3} \right)^N \)

Multiplying both sides of this equation by 3 gives us:

\[ S = 1 - \left( \frac{2}{3} \right)^N \]

Since there is always a positive distance left, the frog will never reach the pond.

Note: As N gets larger and larger (in mathematics we talk about “as N approaches infinity”), note that the distance left gets smaller and smaller (we say that the term being subtracted “approaches 0”), so that the frog gets closer and closer to completing the distance. In calculus these ideas lead to limits.

(This may spark a conversation among your students regarding the fact that eventually, the frog will get close enough to the pond that he can simply step over into the pond because the distance left is so small.)

**Possible Student Misconceptions**

In the blog problem, students may not understand the difference between the total number of followers and the number of new followers. For example, students could follow this line of reasoning:

- Week 1: There are 5 followers.
- Week 2: Each of these 5 told 2 friends, so we have a total of 15 followers.
- Week 3: Each of the 15 followers told 2 friends, so we have a total of 15 + 30 = 45 followers.
- Week 4: Each of the 45 followers told 2 friends, so we have a total of 45 + 90 = 135 followers.
- Etc.

Read the problem carefully. Explain to me how many followers you would have in each of weeks 1, 2, 3, and 4. How did you calculate the number of new followers each week? Does this reflect the directions in the problem?
In this case, the students are assuming that ALL of the followers from the previous week are telling 2 friends, so they are doubling the TOTAL number of followers, not doubling the total number of NEW followers.

In the frog problem, students may calculate each successive hop as being 1/3 of the previous hop rather than 1/3 of the distance left. This reasoning will work if the ratio is ½ but not if the ratio is any other fraction.

<table>
<thead>
<tr>
<th>Entry/Extensions</th>
<th>Assessing and Advancing Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>If students can’t get started….</td>
<td>What information does the problem ask you to put in your table?</td>
</tr>
<tr>
<td></td>
<td>How does making a table help you organize your information?</td>
</tr>
<tr>
<td></td>
<td>Do you see any patterns in your table?</td>
</tr>
<tr>
<td>If students finish early….</td>
<td>Re-work the frog problem using different values for the common ratio. Is there a reasonable value of the common ratio that will allow your frog to reach the pond in a finite number of hops? Why or why not?</td>
</tr>
<tr>
<td></td>
<td>In the blog problem, are there “real-world” limitations to the number of followers Erin’s blog can have? If so, what are those limitations?</td>
</tr>
</tbody>
</table>

Discuss/Analyze

Whole Group Questions

What happens to the number of blog followers in the first problem as N gets bigger and bigger?
What happens to the distance the frog travels in the second problem as N gets bigger and bigger?
How are these problems alike? How are they different?
What is the key piece of information in the problem that determines whether your sum diverges (as in the blog problem) or converges (as in the frog problem)?