K - 5
Mathematics Participant Packet

November 2013
Reflection
What are the advantages of anticipating students’ responses to cognitively demanding tasks during the lesson planning process?
Anticipate Strategies That Students Might Use to Solve the Tasks

Teaching in a manner that productively makes use of students’ ideas and strategies that are generated by high-level tasks is demanding. It requires knowledge of mathematics content, knowledge of student thinking, knowledge of pedagogical “moves” that a teacher can make to lead discussions, and the ability to rapidly apply all of these in specific circumstances (M. Smith & Stein, 2011). To support teachers in this endeavor, Smith and Stein suggested five practices that are intended to make student-centered instruction more manageable. This is done by moderating the degree of improvisation required from the teacher in the midst of a discussion. Rather than providing an instant fix for mathematics instruction, the five practices provide “a reliable process that teachers can depend on to gradually improve their classroom discussions over time” (Stein, Engle, Smith, & Hughes, 2008, p. 335). The first of the five practices is anticipating students’ solutions to a mathematics task. Anticipating requires considering the different ways the task might be solved. This includes anticipating factors such as how students might mathematically interpret a problem, the array of correct and incorrect strategies students might use to solve it, and how those strategies might relate to the goal of the lesson (M. Smith & Stein, 2011). Anticipating can support teachers’ planning by helping them to consider, in advance, how they might respond to the work that students are likely to produce and how they can use those strategies to address the mathematics to be learned.

Leaves and Caterpillar Task

A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars?

Use drawings, words, or numbers to show how you got your answer.

• Solve the task in as many ways as you can, and consider other approaches that you think students might use to solve it.

• Identify errors or misconceptions that you would expect to emerge as students work on this task.
Use this space for the Leaves and Caterpillar Task.
Anticipating involves carefully considering (1) what strategies students are likely to use to approach or solve a challenging mathematical task (e.g., a high-level task), (2) how to respond to the work students are likely to produce, and (3) which student strategies are likely to be most useful in addressing the mathematics to be learned. We will use an adapted version of the vignette Leaves and Caterpillars: The Case of David Crane to illustrate these three points.

Students in Mr. Crane’s fourth-grade class are to solve the following problem: “A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would the students need each day for 12 caterpillars?” David Crane wants his students (1) to recognize that the relationship between the caterpillars and leaves is multiplicative and not additive— that the 2 quantities (leaves and caterpillars) need to grow at a constant rate. (2) He also hopes students will recognize that there are three related strategies for solving the task – unit rate, scale factor and scaling up. These are the targeted learning goals he has for this task.

David began planning the lesson by anticipating how students might solve the task. His first step in the process was to solve the problem himself by using non-procedural methods, since those are the methods to which his students would have access. He considered three general approaches, as shown is Figure 0.2, as reasonable courses of actions. These are unit rate— students might find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times. The scale factor strategy - students might realize that there is a scale factor of 6—he thought several of his students would realize that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves must be 6 times the original amount (5). David also thought his students might use the scaling up strategy by increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until they reached the desired number of caterpillars (12).

David planned to watch and listen during small-group independent work, he wanted to use his observations to decide what and who to highlight during the discussion that follows. He will look for a solution demonstrating unit rate, finding the number of leaves one caterpillar eats and multiply by twelve. This would be a good example of multiplicative reasoning. If a pair of students chose to add the amount for one caterpillar 12 times, this would be additive reasoning and he wants to move them toward a more efficient approach. He planned to ask students, “If the numbers were larger, would this be the best strategy (for instance if there were 500 caterpillars)?” He wants to highlight the scale factor approach, the number of caterpillars is 6 times the original amount, so the number of leaves must be 6 times the original amount (multiplicative reasoning). If students use the scaling up approach, adding 5 leaves and 2 caterpillars until the desired number of caterpillars is reached (additive), although it is a good strategy, he wanted to have a question ready to move them toward multiplicative reasoning as well. David planned to ask students who used this strategy where they had seen these patterns of numbers before (2, 4, 6, 8, …… or 5, 10, 15, 20, 25, ……).

He thought that some of his students might be confused about different aspects of the task, and he wanted to make sure that he was prepared to deal with these issues as they arose. For example, he thought some students may multiply the 5 leaves by the 12 caterpillars and get 60. What the students failed to reason was 2 caterpillars eat 5 leaves, so only 6 caterpillars would be needed to multiply with 5 leaves. Another misconception the students may have is 10 more
Possible Solutions

**Unit Rate**—Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times.

If each of the caterpillars ate $2 \frac{1}{2}$ leaves a day then you multiply

$2 \frac{1}{2} \times 12 = 30$ leaves

**Scale Factor**—Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5).

If it takes 5 leaves for 2 caterpillars, then $2 \times 6 = 12$ caterpillars and $5 \times 6 = 30$ leaves. So it takes 30 leaves and the scale factor is 6.

**Scaling Up**—Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars.

Table (shows relationship between leaves and caterpillars)

<table>
<thead>
<tr>
<th>Leaves</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Caterpillars</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

Fig. 0.2 David Crane’s possible solutions
caterpillars were added in the problem, so 10 more leaves need to be added to give a total of 15 leaves (additive thinking, not multiplicative). Although he wants his students to be able to figure out their errors on their own, he wanted to be ready with some questions that would guide their thinking in the right direction. David planned to ask students, “What is the connection between the leaves and the caterpillars?” Anticipating their strategies in advance he felt would make it possible for him to have a question ready to ask that might help these and other students recognize why this approach, though tempting, doesn’t work.

David wanted to make sure when he got to the end of the lesson, he would have accomplished what he set out to do. (i.e., (1) to recognize that the relationship between the caterpillars and leaves is multiplicative and not additive (2) student work includes examples that connect with known mathematical strategies (e.g., unit rate, scale factor, scaling up) and to do so, he knew he needed to have correct versions of all three representations-modeling, tables, and mathematical reasoning available for discussion. Because he wants to build the discussion around the work that students produced, if at all possible, he decided to keep track of what students were doing as he observes and interacts with them as they work on the task in small groups. To facilitate this process, he made a chart of the strategy used, who and what representations was used, and order with which he wants to discuss the solutions. He thought the chart would help him in planning the class discussion.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Who and What</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scale Factor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scaling up</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Additive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1.1, A chart for monitoring students’ work on the Leaves and Caterpillars task
## “Anticipating” Vignette

Use the numbered lines in the vignette to guide you as you record your notes.

<table>
<thead>
<tr>
<th>What strategies does David anticipate students are likely to use? What aspects of the task does he believe will challenge students?</th>
<th>How does David plan to respond to the work students are likely to produce?</th>
<th>Which student strategies does David identify as being most useful in addressing the mathematical goals?</th>
</tr>
</thead>
</table>
Journal Reflection

What could you do differently in your own practice to improve your ability to anticipate student responses?

[Blank lines]

[Blank lines]
Fig. 2 Thinking Through a Lesson Protocol (TTLP)

**PART 1: SELECTING AND SETTING UP**

**A MATHEMATICAL TASK**

What are your mathematical goals for the lesson (i.e., what do you want students to know and understand about mathematics as a result of this lesson)?

In what ways does the task build on students’ previous knowledge, life experiences, and culture? What definitions, concepts, or ideas do students need to know to begin to work on the task? What questions will you ask to help students access their prior knowledge and relevant life and cultural experiences?

What are all the ways the task can be solved?

- Which of these methods do you think your students will use?
- What misconceptions might students have?
- What errors might students make?

What particular challenges might the task present to struggling students or students who are English language learners? How will you address these challenges?

What are your expectations for students as they work on and complete this task?

- What resources or tools will students have to use in their work that will give them entry into, and help them reason through, the task?
- How will the students work—Independently, in small groups, or in pairs—to explore this task? How long will they work individually or in small groups or pairs? Will students be partnered in a specific way? If so, in what way?
- How will students record and report their work?

How will you introduce students to the activity so as to provide access to all students while maintaining the cognitive demands of the task? How will you ensure that students understand the context of the problem? What will you hear that lets you know students understand what the task is asking them to do?

**PART 2: SUPPORTING STUDENTS’ EXPLORATION OF THE TASK**

As students work independently or in small groups, what questions will you ask to—

- help a group get started or make progress on the task?
- focus students’ thinking on the key mathematical ideas in the task?

- assess students’ understanding of key mathematical ideas, problem-solving strategies, or the representations?
- advance students’ understanding of the mathematical ideas?
- encourage all students to share their thinking with others or to assess their understanding of their peers’ ideas?

How will you ensure that students remain engaged in the task?

- What assistance will you give or what questions will you ask a student (or group) who becomes quickly frustrated and requests more direction and guidance in solving the task?
- What will you do if a student (or group) finishes the task almost immediately? How will you extend the task so as to provide additional challenge?
- What will you do if a student (or group) focuses on nonmathematical aspects of the activity (e.g., spends most of his or her (or their) time making a poster of their work)?

**PART 3: SHARING AND DISCUSSING THE TASK**

How will you orchestrate the class discussion so that you accomplish your mathematical goals?

- Which solution paths do you want to have shared during the class discussion? In what order will the solutions be presented? Why?
- In what ways will the order in which solutions are presented help develop students’ understanding of the mathematical ideas that are the focus of your lesson?
- What specific questions will you ask so that students will—
  1. make sense of the mathematical ideas that you want them to learn?
  2. expand on, debate, and question the solutions being shared?
  3. make connections among the different strategies that are presented?
  4. look for patterns?
  5. begin to form generalizations?

How will you ensure that, over time, each student has the opportunity to share his or her thinking and reasoning with their peers?

What will you see or hear that lets you know that all students in the class understand the mathematical ideas that you intended for them to learn?

What will you do tomorrow that will build on this lesson?


One way to both control teaching with high-level tasks and promote success is through detailed planning prior to the lesson. TTLP is a process that is intended to further the use of cognitively challenging tasks (Smith and Stein 1998).
Use this space to show your solution paths for the task.
### Standard #

<table>
<thead>
<tr>
<th>Task: Setting Goals</th>
</tr>
</thead>
<tbody>
<tr>
<td>What are your mathematical goals for the lesson (i.e., what do you want students to know and understand about mathematics as a result of this lesson)?</td>
</tr>
</tbody>
</table>

#### Mathematical Objective and Goals

<table>
<thead>
<tr>
<th>Task: Anticipating</th>
</tr>
</thead>
<tbody>
<tr>
<td>What are all the ways the task can be solved?</td>
</tr>
<tr>
<td>• Which of these methods do you think students will use?</td>
</tr>
<tr>
<td>• What misconceptions might students have?</td>
</tr>
<tr>
<td>• What errors might students make?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Task: Anticipating Student Responses to Challenging Mathematical Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>As students work independently or in small groups, what questions will you ask to—</td>
</tr>
<tr>
<td>help a group get started or make progress on the task?</td>
</tr>
<tr>
<td>• focus students’ thinking on the key mathematical ideas in the task?</td>
</tr>
<tr>
<td>• assess students’ understanding of key mathematical ideas, problem-solving strategies, or the representations?</td>
</tr>
<tr>
<td>• advance students’ understanding of the mathematical ideas?</td>
</tr>
<tr>
<td>• encourage all students to share their thinking with others or to assess their understanding of their peers’ ideas?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Task: Anticipating How to Respond to the Work Students Are Likely to Produce</th>
</tr>
</thead>
<tbody>
<tr>
<td>Which student strategies do you identify as being most useful in addressing the mathematical goals?</td>
</tr>
</tbody>
</table>
### Summary of Standards for Mathematical Practice

#### 1. Make sense of problems and persevere in solving them.
- Interpret and make meaning of the problem to find a starting point. Analyze what is given in order to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, “Does this make sense?”

#### 2. Reason abstractly and quantitatively.
- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attends to the meaning of quantities, not just how to compute them.

#### 3. Construct viable arguments and critique the reasoning of others.
- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

#### 4. Model with mathematics.
- Understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the mathematics they know to solve everyday problems.
- Are able to simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, “How can I represent this mathematically?”

### Questions to Develop Mathematical Thinking

#### 1. Make sense of problems and persevere in solving them.
- How would you describe the problem in your own words?
- How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem?
- Describe the relationship between the quantities.
- Describe what you have already tried. What might you change?
- Talk me through the steps you’ve used to this point.
- What steps in the process are you most confident about?
- What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin?
- How else might you organize...represent... show...?

#### 2. Reason abstractly and quantitatively.
- What do the numbers used in the problem represent?
- What is the relationship of the quantities?
- How is _______ related to ________?
- What is the relationship between _______ and ________?
- What does _______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might we use to find a solution?
- How did you decide in this task that you needed to use...?
- Could we have used another operation or property to solve this task? Why or why not?

#### 3. Construct viable arguments and critique the reasoning of others.
- What mathematical evidence would support your solution?
- How can we be sure that...? / How could you prove that...?
- Will it still work if...?
- What were you considering when...?
- How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn’t it work? Would it ever work? Why or why not?
- What is the same and what is different about...?
- How could you demonstrate a counter-example?

#### 4. Model with mathematics.
- What number model could you construct to represent the problem?
- What are some ways to represent the quantities?
- What is an equation or expression that matches the diagram, number line, chart, table...
- Where did you see one of the quantities in the task in your equation or expression?
- How would it help to create a diagram, graph, table...
- What are some ways to visually represent...
- What formula might apply in this situation?
<table>
<thead>
<tr>
<th>Standards for Mathematical Practice</th>
<th>Questions to Develop Mathematical Thinking</th>
</tr>
</thead>
</table>
| **5. Use appropriate tools strategically.**<br>  
  - Use available tools recognizing the strengths and limitations of each.<br>  - Use estimation and other mathematical knowledge to detect possible errors.<br>  - Identify relevant external mathematical resources to pose and solve problems.<br>  - Use technological tools to deepen their understanding of mathematics.  |
| What mathematical tools could we use to visualize and represent the situation?  
What information do you have?  
What do you know that is not stated in the problem?  
What approach are you considering trying first?  
What estimate did you make for the solution?  
In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative?  
Why was it helpful to use...?  
What can using a ______ show us that _____ may not?  
In what situations might it be more informative or helpful to use...?  |
| **6. Attend to precision.**<br>  
  - Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.<br>  - Understand the meanings of symbols used in mathematics and can label quantities appropriately.<br>  - Express numerical answers with a degree of precision appropriate for the problem context.<br>  - Calculate efficiently and accurately.  |
| What mathematical terms apply in this situation?  
How did you know your solution was reasonable?  
Explain how you might show that your solution answers the problem.  
What would be a more efficient strategy?  
How are you showing the meaning of the quantities?  
What symbols or mathematical notations are important in this problem?  
What mathematical language..., definitions..., properties can you use to explain...?  
How could you test your solution to see if it answers the problem?  |
| **7. Look for and make use of structure.**<br>  
  - Apply general mathematical rules to specific situations.<br>  - Look for the overall structure and patterns in mathematics.<br>  - See complicated things as single objects or as being composed of several objects.  |
| What observations do you make about...?  
What do you notice when...?  
What parts of the problem might you eliminate..., simplify...?  
What patterns do you find in...?  
How do you know if something is a pattern?  
What ideas that we have learned before were useful in solving this problem?  
What are some other problems that are similar to this one?  
How does this relate to...?  
In what ways does this problem connect to other mathematical concepts?  |
| **8. Look for and express regularity in repeated reasoning.**<br>  
  - See repeated calculations and look for generalizations and shortcuts.<br>  - See the overall process of the problem and still attend to the details.<br>  - Understand the broader application of patterns and see the structure in similar situations.<br>  - Continually evaluate the reasonableness of their intermediate results  |
| Explain how this strategy work in other situations?  
Is this always true, sometimes true or never true?  
How would we prove that...?  
What do you notice about...?  
What is happening in this situation?  
What would happen if...?  
Is there a mathematical rule for...?  
What predictions or generalizations can this pattern support?  
What mathematical consistencies do you notice?  |